Appendix D
General Analytical Solutions for Application to Pumping Test Data

This appendix contains equations dealing with the effects of pumping on aquifers, both confined and unconfined, as well as the analysis of pump test results.

D-1. Flow Equations for Aquifer Pumping

a. *Steady-state solution in a confined aquifer*. For a pumping well similar to that shown in Figure D-1, the magnitude of radial flow to a well is calculated by the Thiem equation:

$$Q = 2\pi bK\frac{h_i - h_w}{\ln\left(\frac{r_i}{r_w}\right)} \qquad \text{(D-1)}$$

where

Q = flow into the well [L^3/T]

b = aquifer thickness [L]

K = hydraulic conductivity [L/T]

h_i = head at observation well (distance r_i away) [L]

h_w = head at well [L]

r_i = radius from center of well to observation well [L]

r_w = radius of well [L]

In the design of pump-and-treat systems, it is often desired to estimate the lateral width of the influence of the well. The capture zone of a pumping well under steady-state conditions and assuming uniform flow can be estimated by:

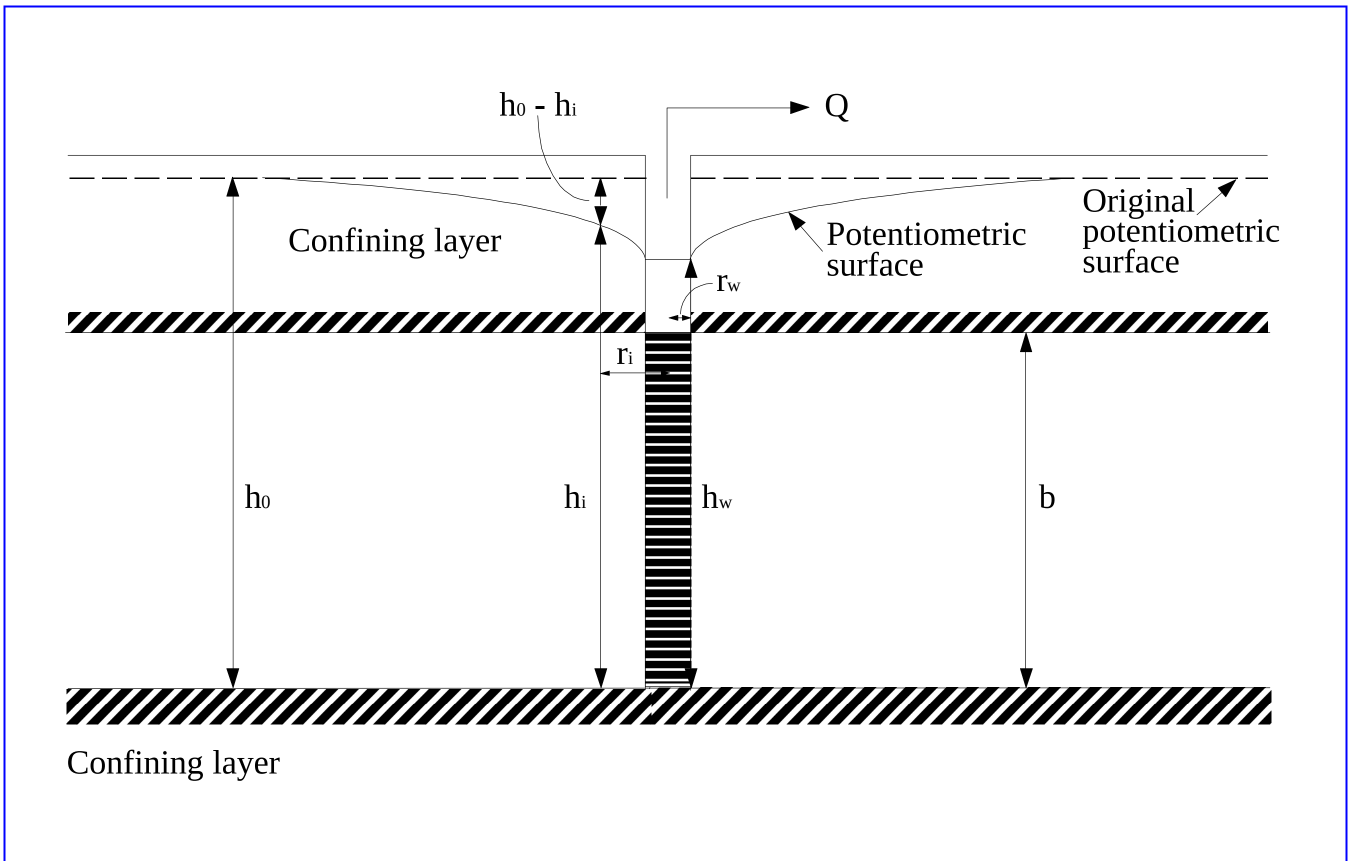

Figure D-1. Parameters used in solution for radial flow to a well in a confined aquifer

$$y_c = \frac{Q}{vb} \qquad \text{(D-2)}$$

where

y_c = capture zone perpendicular to antecedent flow direction [L]

Q = flow into the well [L^3/T]

v = specific discharge = $-Kdh/dl$ [L/T]

b = aquifer thickness [L]

b. *Steady-state solution in an unconfined aquifer.*

(1) The solution of the Laplace equation for an unconfined aquifer is similar to that for the confined aquifer, except that change in aquifer thickness must be addressed. Figure D-2 presents a graphical representation of flow to a well in an unconfined aquifer. The magnitude of radial flow can be calculated by:

$$Q = \frac{\pi K(h_2^2 - h_1^2)}{\ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(D-3)}$$

where

Q = flow to well

h_1 = head at point 1 [L]

h_2 = head at point 2 [L]

r_1 = radial distance from well to point 1 [L]

r_2 = radial distance from well to point 2 [L]

Assumptions (known as the Dupuit assumptions) inherent in this equation are: 1) flow lines are horizontal and equipotentials are vertical; and 2) the hydraulic gradient is equal to the slope of the water table and is invariant with depth. Equation D-3 is useful for approximating the hydraulic conductivity of an aquifer.

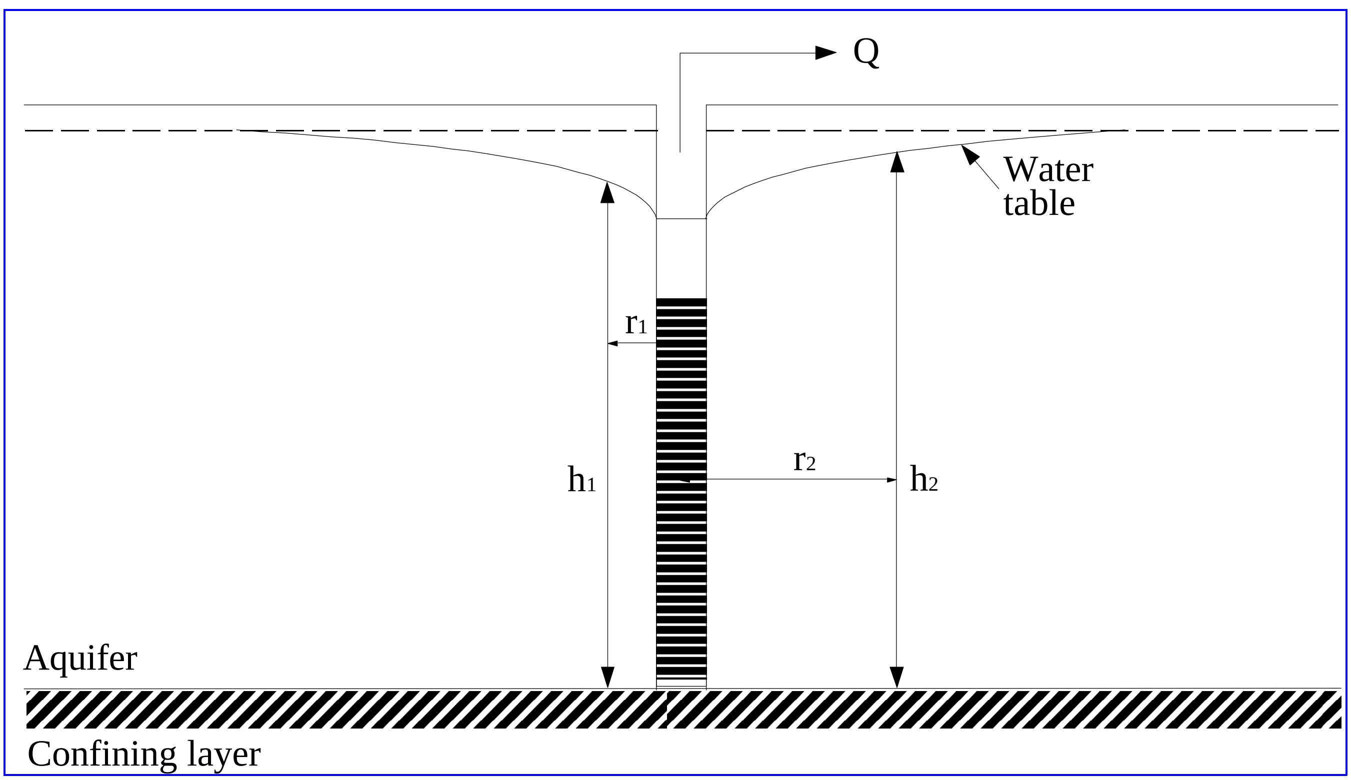

Figure D-2. Parameters used in solution for radial flow to a well in an unconfined aquifer

(2) If observation wells are not available, the field application for Equations D-1 and D-3 can be simplified by allowing r_1 to equal the radius of the well (r_w), and r_2 to be approximated by the radius of influence of the well, i.e., the radial distance at which drawdown approaches zero.

c. Unsteady flow solution in confined aquifers.

(1) The solution of flow to a well under transient (non-steady) conditions is complicated. Assumptions used in simplification are: a) that the aquifer is isotropic, homogeneous, and of infinite areal extent; b) the well fully penetrates the aquifer; c) the flow is horizontal everywhere within the aquifer; d) the well diameter is so small that storage within the well is negligible, and; e) water pumped from the well is discharged immediately with decline of piezometric head. Theis, in 1931, gave the following solution for this problem:

$$s = \frac{Q}{4\pi T} W(u) \qquad \text{(D-4)}$$

where

s = drawdown [L]

(2) Drawdown is defined as the distance from the original piezometric surface to the new surface at a point a distance r from the center of the pumping well.

$W(u)$ = well function term, which is defined as:

$$W(u) = \int_u^\infty \frac{e^{-u}}{u}\, du \qquad \text{(D-5)}$$

$$u = \frac{r^2 S}{4Tt} \qquad \text{(D-6)}$$

where

r = radial distance in feet [L]

S = storativity coefficient [dimensionless]

t = duration of pumping [T]

T = transmissivity [L^2/T]

(3) Table D-1 is a tabulation of $W(u)$ given u, which is easily calculated if S and T are known. Unfortunately, these aquifer parameters are usually unknown and pumping tests must be performed. By observing

Table D-1
Tabulation of W(u) Values for Use in Theis Equation

	Values of *W*(*u*) for values of *u*								
u	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
x 1	0.219	0.049	0.013	0.0038	0.0011	0.00036	0.00012	0.000038	0.000012
x 10-1	1.82	1.22	0.91	0.70	0.56	0.45	0.37	0.31	0.26
x 10-2	4.04	3.35	2.96	2.68	2.47	2.30	2.15	2.03	1.92
x 10-3	6.33	5.64	5.23	4.95	4.73	4.54	4.39	4.26	4.14
x 10-4	8.63	7.94	7.53	7.25	7.02	6.84	6.69	6.55	6.44
x 10-5	10.94	10.24	9.84	9.55	9.33	9.14	8.99	8.86	8.74
x 10-6	13.24	12.55	12.14	11.85	11.63	11.45	11.29	11.16	11.04
x 10-7	15.54	14.85	14.44	14.15	13.93	13.75	13.60	13.46	13.34
x 10-8	17.84	17.15	16.74	16.46	16.23	16.05	15.90	15.76	15.65
x 10-9	20.15	19.45	19.05	18.76	18.54	18.35	18.20	18.07	17.95
x 10-10	22.45	21.76	21.35	21.03	20.84	20.66	20.50	20.37	20.25
x 10-11	24.75	24.06	23.65	23.36	23.14	22.96	22.81	22.67	22.55
x 10-12	27.05	26.36	25.96	25.67	25.44	25.26	25.11	24.97	24.86
x 10-13	29.36	28.66	28.26	27.97	27.75	27.56	27.41	27.28	27.16
x 10-14	31.66	30.97	30.56	30.27	30.05	29.87	29.71	29.58	29.46
x 10-15	33.96	33.27	32.86	32.58	32.35	32.17	32.02	31.88	31.76

the drawdown over a period of time, the Theis equation can be solved for S and T. This solution is not explicit and the solution is usually obtained graphically.

d. *Unsteady flow in an unconfined aquifer.* In practice, drawdowns in unconfined aquifers can be significant and the assumption that water released from storage is discharged immediately with decline of piezometric head is breached. Unconfined aquifers generally exhibit a behavior called delayed yield in which water is released from storage and specific yield at a time after pumping has started. Flow to a pumping well in an unconfined (phreatic) aquifer occurs in three phases. In the first phase, pumping has just started, and a phreatic aquifer behaves like a confined aquifer. Water is derived from storage i.e., expansion of water and compression of the aquifer. The time-drawdown plot for this phase emulates the Theis curve. In the second phase, the phenomenon of delayed yield occurs. During this phase, water remaining in the pores is drained by gravity (specific yield). This gravity drainage replenishes that portion of the aquifer supplying water to the well, resulting in a reduction in the rate of drawdown over the first phase. This appears as the time-drawdown plot flattens in response to the secondary source of water. In the third phase, the rate of drawdown and the time-drawdown again emulate a Theis curve. The duration of either of the first two phases is a function of the ratio of storage (S) to specific yield (S_y). If this ratio is in the range of 10^{-1} and 10^{-2}, it is an indication that S is relatively large and that one can anticipate a significant first phase. The type of materials you would expect to exhibit this behavior are sandy silts, silty-, clayey-, or fine-grained sands. When S/S_y is in the range of 10^{-4} to 10^{-6} it is an indication that S_y is relatively large and that one can anticipate a significant second phase. The type of materials you would expect to exhibit this behavior are clean sands and gravels. In addition to S/S_y, the geometry of the time-drawdown graph can be affected by the location of the observation well(s). As the distance to an observation well increases, the effects of S diminish.

D-2. Analysis of Pump Test Results

a. *Theis method.* As discussed in Section D-1, the Theis equation allows for the determination of the hydraulic characteristics of an aquifer (transmissivity and storativity) before the development of new steady-state conditions resulting from pumping. The assumptions inherent in the Theis equation include: transmissivity is constant to the extent of the cone of depression, well discharge is derived entirely from aquifer storage and is discharged instantaneously with decline in head, the discharging well penetrates the entire thickness of the aquifer, and well diameter is small in comparison with the pumping rate. These assumptions are best met by confined aquifers at sites remote from their boundaries. The Theis equation is of a form which cannot be solved directly, and is solved through the use of a graphic method of solution called type curves. Analysis of aquifer-test data involves plotting the test data (drawdown versus time) on logarithmic graph paper and aligning this curve with a corresponding type curve from which values of transmissivity and storativity can be calculated. The Theis equation can be used for unconfined aquifers under the following two considerations: (1), if the aquifer is relatively fine-grained, water is not released instantaneously with the decline in head; thus, the value of storativity determined from a short-period test may be too small; and (2) the effect of dewatering the aquifer decreases aquifer thickness and thus transmissivity. This dewatering effect can be addressed by the following equation:

$$s' = s - (s^2/2b) \qquad \text{(D-7)}$$

where

s = observed drawdown in the unconfined aquifer [L]

b = aquifer thickness [L]

s' = drawdown that would have occurred if the aquifer was confined [L]

b. *Cooper-Jacob method.* The Cooper-Jacob method, developed by C. E. Jacob and H. H. Cooper in 1946, simplifies the Theis method. This method uses the fact that after a sufficiently long pumping time or at a sufficient distance from the well, the test data tend to form a straight line when plotted on a semilog scale. The slope of the line formed allows the calculation of transmissivity (T) and storativity (S). The key additional assumption in the Cooper-Jacob method is that it is only applicable to that portion of the cone of depression where steady-state conditions prevail, or to the entire cone only after steady-state conditions have developed. This assumption is necessary because after steady-state conditions have developed, the drawdowns at an observation well begin to fall along a straight line on a semi-log graph. The slope and zero drawdown line intercept of this line can be entered into the following analytical equations:

$$T = \frac{2.3Q}{4\pi \Delta s} \qquad \text{(D-8)}$$

$$S = \frac{2.25Tt_0}{r^2} \qquad \text{(D-9)}$$

where

Q = pumping rate [L^3/T]

Δs = drawdown across one log cycle [L]

t_0 = time at the point where a straight line intersects the zero-drawdown line [T]

r = distance from the pumping well to the observation well [L]

c. Distance-drawdown analysis. In the Jacob distance-drawdown method, drawdowns at a specific moment, from at least three observation wells located at different distances from the pumping well are plotted on a drawdown (arithmetic) and distance (logarithmic) axis. If the Theis assumptions and Jacob limitations are satisfied, a straight line will be produced. The slope of this line is proportional to transmissivity and pumping rate. Storativity can then be computed as a function of transmissivity, time, and the value of the intercept at the point of zero drawdown.

$$T = \frac{2.3Q}{2\pi \Delta s} \qquad \text{(D-10)}$$

$$S = \frac{2.25Tt}{r_0^2} \qquad \text{(D-11)}$$

where

Q = pumping rate [L^3/T]

t = time at which the drawdowns were measured [T]

Δs = drawdown across one log cycle [L]

r_0 = radial distance from the pumping well to the point where there is zero drawdown [L]

d. Corrections for partial penetration. As previously stated, the Theis method assumes that pumping wells fully penetrate the aquifer and all releases from storage are derived directly and solely from the aquifer being pumped. Partial penetration of the well into the aquifer causes vertical gradients of head to occur. These vertical gradients in the vicinity of the well violate a main assumption inherent in the fully penetrating well solution. When the well only partially penetrates the aquifer, the average flow path length is increased so that a greater resistance to flow is encountered. The relationship between flow Q and drawdown s between the partially penetrating (subscript p) and fully penetrating well is: if $Q_p = Q$, then $s_p > s$; and if $s_p = s$, then $Q_p < Q$. The effect of partial penetration is negligible on the flow pattern and drawdown if the radial distance from the well to a point is greater than 7.5 times the saturated thickness b of the aquifer.

e. Vertical leakance. In the development of the Theis equation for the analysis of pumping-test data, it was assumed that all water discharged from a pumping well was derived instantaneously from storage in the aquifer. Therefore, in the case of a confined aquifer, at least during the period of the test, the flow of water across the confining beds is considered negligible. This assumption is often valid for many confined aquifers. Many other aquifers, however, are bound by leaky confining beds which transmit water to the well other than that specified by the Theis equation. The analysis of aquifer tests conducted in these environments requires the implementation of algorithms that have been developed for semi-confined aquifers.

f. References. The previously mentioned methods of analysis cover only a small fraction of the analytical methods available for a wide range of geologic/aquifer settings. It is recommended that the reader obtain a text such as Dawson and Istok (1991), Driscoll (1986), Kruseman and DeRidder (1983), and Walton (1987) for a comprehensive treatment of the subject matter.